Semester I Unit-05

Theory of equation

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[5.1.8] Theorem :- To Prove "In an equation with real coefficients, imaginary roots occur in conjugate pairs."

Proof :—

Let $\alpha+i\beta$ be a roots of polynomial eqn $f(x)=0$

Then we have to prove that $\alpha-i\beta$ is also a root of the equation.

We divide the polynomial $f(x)$ by

$$\{x-(\alpha+i\beta)\}\{x-(\alpha-i\beta)\}$$

i.e. $\{(x-\alpha)-i\beta\}\{(x-\alpha)+i\beta\} = (x-\alpha)^2+\beta^2$

Let the quotient be Q and the remainder be $Rx+R'$.

Then, we have

$$f(x) = \{(x-\alpha)^2+\beta^2\}Q+(Rx+R') \quad \text{—①}$$

Now, since $(\alpha+i\beta)$ is a roots of the equation

$$\therefore f(\alpha+i\beta)=0 \quad \text{—②}$$

Hence, Putting $x=\alpha+i\beta$ in ①, we get

$$f(\alpha+i\beta) = \{(\alpha+i\beta-\alpha)^2+\beta^2\}.Q+R(\alpha+i\beta)+R'$$

$$\Rightarrow f(\alpha+i\beta) = 0.Q+R(\alpha+i\beta)+R'$$

$$\therefore 0 = 0+R(\alpha+i\beta)+R' \quad \{\text{from ②}\}$$

$$\Rightarrow R\alpha+iR\beta+R'=0$$

$$\Rightarrow (R\alpha+R')+iR\beta=0=0+i.0$$

$$\Rightarrow R\alpha+R'=0 \text{ and } R\beta=0$$

$0.\alpha+R'=0$ | $R=0$ because, $\beta\neq0$

$R'=0$

Hence, from ① $f(x) = \{(x-\alpha)^2+\beta^2\}.Q$

$$=\{(x-\alpha+i\beta)(x-\alpha-i\beta)\}Q$$

$$=\{x-(\alpha-i\beta)\}\{x-(\alpha+i\beta)\}Q$$

This shows that $\{x-(\alpha-i\beta)\}$ is a factor of $f(x)$.

$\therefore$ $x=\alpha-i\beta$ is a root of $f(x)=0$

Proved.

~~# 5.1.9 L~~

Note: Cor:- Every equation of odd degree has at least one real root.

5.1.9 Theorem:- To Prove, "In an equation with rational coefficients irrational root occur in conjugate pair." (same as 5.1.8)

Proof:-

Let $\alpha+\sqrt{\beta}$ be a root of the eq^n $f(x)=0$.

Then, we have to prove that

$\alpha-\sqrt{\beta}$ is also a root of equation $f(x)=0$.

We divide $f(x)$ by $\{x-(\alpha+\sqrt{\beta})\}\{x-(\alpha-\sqrt{\beta})\}$

i.e. $\{(x-\alpha)-\sqrt{\beta}\}\{(x-\alpha)+\sqrt{\beta}\}$

i.e. $(x-\alpha)^2-\beta$.

Let the quotient be Q and the remainder be $(Rx+R')$.

Then, we have,

$$f(x)=\{(x-\alpha)^2-\beta\}Q+(Rx+R')$$

$$=\{x-(\alpha+\sqrt{\beta})\}\{x-(\alpha-\sqrt{\beta})\}Q+(Rx+R') \quad \text{--- (1)}$$

Now, Since, $\alpha+\sqrt{\beta}$ is a root of equation $f(x)=0$,

$\therefore$ $f(\alpha+\sqrt{\beta})=0$ --- (2)

Putting $x=\alpha+\sqrt{\beta}$ in (1), we get

$$\cancel{f(\alpha+\sqrt{\beta})=0\cdot Q+R(\alpha+\sqrt{\beta})+R'}$$

$$0=0+R(\alpha+\sqrt{\beta})+R' \quad \{\text{from (2)}\}$$

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$(R\alpha + R') + R\sqrt{\beta} = 0 \cancel{= 0 + 0\sqrt{\beta}}$

$\therefore \; R\alpha + R' = 0$ and $R\sqrt{\beta} = 0$

$0 \cdot \alpha + R' = 0$ | $\boxed{R = 0}$ because, $\beta \neq 0$

$\boxed{R' = 0}$

Hence, from ①

$$f(x) = \{x - (\alpha + \sqrt{\beta})\}\{x - (\alpha - \sqrt{\beta})\}\phi$$

This shows that $\{x - (\alpha - \sqrt{\beta})\}$ is a factor of $f(x)$.
Consequently, $x = \alpha - \sqrt{\beta}$ is a root of eqn $f(x) = 0$

Proved

$\boxed{5.1.10}$ Synthetic division:-

Let $f(x) \equiv a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1}x + a_n$ — ①
is divided by binomial $(x - h)$.

$$\begin{array}{c|cccccc} h & a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ & & b_0 h & b_1 h & \dots & \dots & b_{n-1} h \\ \hline & b_0 & b_1 & b_2 & & b_{n-1} & R \end{array}$$

$\therefore$ Quotient $= b_0 x^{n-1} + b_1 x^{n-2} + \dots + b_{n-1}$

Remainder $= R$

Example: Find quotient and remainder when $x^3 - 5x^2 + 6x + 7$ is divided by $(x-5)$

$f(x) = x^3 - 5x^2 + 6x + 7$, $g(x) = x - 5$ $\quad \therefore x - 5 = 0$, $x = 5$

Ans

$$\begin{array}{c|cccc} & 1 & -5 & 6 & 7 \\ 5 & & 5 & 0 & 30 \\ \hline & 1 & 0 & 6 & \boxed{37} \end{array} \longrightarrow \text{Remainder.}$$

Quotient $= 1 \cdot x^2 + 0 \cdot x + 6 = x^2 + 6$

Remainder $= 37$